# BOUNDARY MEASURES OF MARKOV CHAINS

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ABSTRACT

It is known that each Markov chain has associated with it a polytope and a family of Markov measures indexed by the interior points of the polytope. Measure-preserving factor maps between Markov chains must preserve the associated families. In the present paper, we augment this structure by identifying measures corresponding to points on the boundary of the polytope. These measures are also preserved by factor maps. We examine the data they provide and give examples to illustrate the use of this data in ruling out the existence of factor maps between Markov chains.

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## 1. Introduction

An irreducible matrix of nonnegative polynomials determines a shift of finite type with weights on its periodic orbits. We study weight-preserving factor maps between such systems, corresponding to measure-preserving factor maps between Markov chains. Letting $\mathbb{R}^{++}$ denote the positive reals, this appropriation of polynomial matrices for the study of Markov chains stems from the fact a family of Markov measures indexed by $(\mathbb{R}^{++})^k$ for a suitable $k$ may be invariantly associated to a single Markov measure [7]. We saw in [3] that the interior of the weight-per-symbol polytope of [2] may be naturally identified with $(\mathbb{R}^{++})^k$; every element of the family of Markov measures thus corresponds to an interior point of the polytope.

In the present paper we identify measures corresponding to points on the boundary of the polytope. These boundary measures are obtained by taking limits of interior measures. Though interior measures are irreducible, boundary measures are often not. However, the support of a boundary measure is a nonwandering shift of finite type and we identify the irreducible components of the support as specific selections from the boundary components defined in [2]. The restriction of the boundary measure to one of these components is, after normalization, the naturally induced Markov measure of [1, 2]. Hence, a boundary measure is characterized by the list of factors by which it scales the Markov measures of the boundary components, data that is conveniently stored in a probability vector. We give examples to illustrate the use of this data in ruling out the existence of factor maps between Markov chains.

In order to define a boundary measure at a point, we need the beta function to be analytic at that point. We start by summarizing the definitions and facts we need from [2, 7], including a result showing that the beta function is analytic at most boundary points. Then we give the main results, whose proofs use tools of [7] involving connecting paths between boundary components. We end the paper with examples.

## 2. Markov chains and boundaries

Let $R = \mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$ be the ring of Laurent polynomials with integral coefficients in the variables $x_1, \ldots, x_k$ and let $R^+ = \mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ be its positive cone consisting of polynomials with positive coefficients and the zero polynomial. For a monomial $m = x_1^{w_1} x_2^{w_2} \cdots x_k^{w_k}$ of $R$, we will write $\log(m) = (w_1, w_2, \ldots, w_k)$.

Let $A$ be an irreducible matrix over $R^+$. Whenever we substitute positive numbers $a_1, \dots, a_k$ for $x_1, \dots, x_k$ we get an irreducible nonnegative real-valued matrix; let $\beta_A(a_1, \dots, a_k)$ be the maximum eigenvalue of $A(a_1, \dots, a_k)$ furnished by the Perron-Frobenius theorem [6]. The resulting function $\beta = \beta_A \colon (\mathbb{R}^{++})^k \to \mathbb{R}^{++}$ is the **beta function** of $A$. Clearly $\beta$ is an algebraic function. Corresponding to $\beta$ there are left and right eigenvectors, $l = l_A$ and $r = r_A$, of $A$ over $R[\beta]$. For example we could take $l$ and $r$ to be a row and a column, respectively, of the adjoint of $\beta I - A$. Make sure that the entries of $l$ are coprime; likewise for $r$. Let us agree to refer to the elements of the indexing set of $A$ as its **states**.

The matrix $A$ defines a weighted graph $G(A)$: The vertices of $G(A)$ are the states of $A$. For states $I$, $J$, the corresponding entry $A(I, J)$, if nonzero, is a sum of monomials, possibly with multiplicity. For each monomial in this sum $G(A)$ has an edge from $I$ to $J$ with that monomial as its weight. We denote the weight of an edge $e$ by $\mathrm{wt}_A(e)$. The **weight** of a path $e_1 \cdots e_n$ in $G(A)$ is $\mathrm{wt}_A(e_1 \cdots e_n) = \prod_{i=1}^n \mathrm{wt}_A(e_i)$. The **weight-per-symbol (wps)** of a cycle $\gamma = e_1 \cdots e_n$ of $G(A)$ is defined to be

$$\mathrm{wps}_A(\gamma) = \frac{1}{n} \log(\mathrm{wt}_A(\gamma)) \in \mathbb{Q}^k.$$

From the weighted graph $G(A)$ we obtain $\Sigma_A$, a shift of finite type with wps's assigned to its periodic orbits. In fact, $\Sigma_A$ represents a shift of finite type with a family of Markov measures $\mu_{A(a_1,\dots,a_k)}$ indexed by $(a_1, \dots, a_k) \in (\mathbb{R}^{++})^k$: Let $T = \sum_I l(I)r(I)$, the sum being over all states $I$ of $A$. Considering a cylinder set

$$[e_0, e_1, \dots, e_m]_h = \{(y_n) \in \Sigma_A \colon y_h = e_0, y_{h+1} = e_1, \dots, y_{h+m} = e_m\},$$

identify the starting state $I$ of $e_0$ and the terminal state $J$ of $e_m$. The measure

$$\mu_{A(a_1,\dots,a_k)}([e_0, e_1, \dots, e_m]_h)$$

then equals the value at $(a_1, \dots, a_k)$ of

$$l(I)\, \mathrm{wt}_A(e_0 e_1 \cdots e_m) r(J)/T\beta^{m+1}.$$

The motivation for this framework stems from the fact that for a number of coding problems the family $\mu_{A(a_1,\dots,a_k)}$ may be invariantly associated with any one of its generic elements; this is explained in detail in §2 of [7].

According to [2], the convex hull, taken in $\mathbb{Q}^k$, of

$$\{\mathrm{wps}_A(\gamma)\colon \gamma \text{ cycle of } G(A)\}$$

is a polytope, which we call the **weight-per-symbol polytope** of $A$ and denote $W(A)$. Let $F$ be a face of $W(A)$. Consider the subgraph of $G(A)$ obtained by keeping only those edges $e$ that lie on a cycle $\gamma$ with $\mathrm{wps}_A(\gamma) \in F$. By [2], this subgraph is nonwandering in the sense that whenever an edge of it leads from a state $I$ to a state $J$ then, in fact, $I$ and $J$ lie in the same irreducible component. The irreducible components of this subgraph, we call the **$F$-components** of $A$.

Before going any further, we take advantage of the fact that a change of variable and division by a monomial allow us to view $F$ very concretely and make the following standing assumption. (See the discussion preceding (11) of [7].)

Standing Assumption. We have $W(A) \subset \{(u_1,\ldots,u_k) \in \mathbb{Q}^k\colon u_k \geq 0\}$ and $F = \{(u_1,\ldots,u_k) \in \mathbb{Q}^k\colon u_k = 0\} \cap W(A)$.

Let $A_{F,1}, A_{F,2}, \ldots, A_{F,l}$ be the irreducible submatrices of $A$ representing the $F$-components of $A$. One consequence of our standing assumption is that each $W(A_{F,i})$ is contained in $\{(x_1,\ldots,x_k) \in \mathbb{Q}^k\colon x_k = 0\}$ and, consequently, that $\beta_{A_{F,i}}$ does not depend on $x_k$. Write $y = x_k$. We extend $\beta$ by putting, for $a_1,\ldots,a_{k-1} > 0$,

$$\beta(a_1,\ldots,a_{k-1},0) = \max\{\beta_{A_{F,i}}(a_1,\ldots,a_{k-1})\colon 1 \leq i \leq l\}.$$

Then $\beta$ is continuously defined on $\{(x_1,\ldots,x_k) \in \mathbb{R}^k\colon x_1,\ldots,x_{k-1} > 0, x_k \geq 0\}$, and analytic on $(\mathbb{R}^{++})^k$. It may fail to be analytic at some points of the form $(a_1,\ldots,a_{k-1},0)$. We will recall from [7] a result related to this. Let $d$ be the degree of $\beta$ over $R$. Let $y = z^{d!}$ and define, for $x_1,\ldots,x_{k-1} > 0$, $z \geq 0$,

$$\widetilde{\beta}(x_1,\ldots,x_{k-1},z) = \beta(x_1,\ldots,x_{k-1},z^{d!}).$$

Let

$$D = \{(a_1,\ldots,a_{k-1}) \in (\mathbb{R}^{++})^{k-1}\colon \widetilde{\beta} \text{ is analytic at } (a_1,\ldots,a_{k-1},0)\}.$$

For a subset $S$ of $\mathbb{R}^{k-1}$, we shall also write $S$ for the set $S \times \{0\} \subset \mathbb{R}^k$; it will always be clear from the context whether we are thinking of $S$ as a subset of $\mathbb{R}^{k-1}$

or of $\mathbb{R}^k$. We similarly identify the points $(a_1,\ldots,a_{k-1})$ and $(a_1,\ldots,a_{k-1},0)$. Note that

$$\widetilde{\beta}(a_1,\ldots,a_{k-1})=\beta(a_1,\ldots,a_{k-1}).$$

It was shown in [7] that there exists a proper variety $V\subset\mathbb{R}^{k-1}$ such that $(\mathbb{R}^{++})^{k-1}\setminus V\subset D$. Consequently, $D$ is an open dense subset of $(\mathbb{R}^{++})^{k-1}$ with finitely many connected components. Also by [7], for a connected component $D_0$ of $D$ and an $F$-component $A_{F,i}$ of $A$, one of two possibilities must hold. Either $\beta_{A_{F,i}}\equiv\beta$ on $D_0$, in which case we call $A_{F,i}$ a **principal component (for $D_0$)**, or $\beta_{A_{F,i}}<\beta$ on $D_0$, in which case we call $A_{F,i}$ a **nonprincipal component (for $D_0$)**.

Finally, let $B$ also be an irreducible matrix over $R^+$. By a **factor map** $\phi\colon\Sigma_A\to\Sigma_B$ we shall mean a bounded-to-one shift-commuting continuous surjection between the shifts of finite type that is wps-preserving in the sense that $\mathrm{wps}_B(\phi(\gamma))=\mathrm{wps}_A(\gamma)$ for every periodic orbit $\gamma$ of $\Sigma_A$. When such a map exists, $\beta_A=\beta_B$, $W(A)=W(B)$ and $F$-components of $A$ cover those of $B$. As already mentioned, the wps-preserving condition corresponds to the requirement that $\phi$ preserve the related Markov measures.

Details of the above definitions and facts may be found in [2] and [7].

## 3. Boundary measures

We work on a connected component $D_0$ of $D$. Recall that $d$ is the degree of $\beta$ and $y=x_k=z^{d!}$, and write $\widetilde{R}=\mathbb{Z}[x_1^{\pm},\ldots,x_{k-1}^{\pm},z^{\pm}]$. Let $f\in\widetilde{R}[\widetilde{\beta}]$. For any $(a_1,\ldots,a_{k-1})\in D_0$, the function $f$ has an expansion

$$z^{n_0}\sum_{w\in(\mathbb{Z}^+)^k}c_w(x_1-a_1)^{w_1}\cdots(x_{k-1}-a_{k-1})^{w_{k-1}}z^{w_k}$$

with $c_w\in\mathbb{R}$ and $n_0\in\mathbb{Z}$. The series is absolutely convergent in a neighbourhood $\mathcal{O}$ of $(a_1,\ldots,a_{k-1},0)$ in $\mathbb{C}^k$ and converges to $f(x_1,\ldots,x_{k-1},z)$ on $\mathcal{O}\cap(\mathbb{R}^+)^k$. Rearranging the terms and using analytic continuation, we write

$$f(x_1,\ldots,x_{k-1},z)=\sum_{n=n_0}^{\infty}f_n(x_1,\ldots,x_{k-1})\,z^n$$

with $n_0\in\mathbb{Z}$, $f_{n_0}$ nontrivial, each $f_n$ analytic on $D_0$ and the (Laurent) series absolutely convergent on a neighbourhood of $D_0$ in $\mathbb{C}^k$ (except, when $n_0<0$, on $D_0$). As in [7], we define $\delta(f) = n_0$ and $f_F = f_{n_0} z^{n_0}$. For the zero function $f = 0$, we take $\delta(f) = \infty$ and $f_F = 0$. Since $y = z^{d!}$, the ring $R[\beta]$ is embedded in $\widetilde{R}[\widetilde{\beta}]$, which gives meaning to $f_F$ and $\delta_F$ for $f \in R[\beta]$. Note that, as a result of our standing assumption, $\delta(\beta) = 0$ and

$$\beta_F(x_1, \ldots, x_{k-1}) = \beta(x_1, \ldots, x_{k-1}, 0) = \widetilde{\beta}(x_1, \ldots, x_{k-1}, 0) > 0$$

on $D_0$. Recall that $l = l_A$ and $r = r_A$ denote left and right coprime eigenvectors over $R[\beta]$.

LEMMA 1: *If two states $I$, $J$ of $A$ belong to the same $F$-component of $A$ then*

$$\delta(l(I)\, r(I)) = \delta(l(J)\, r(J)).$$

*Proof:* Since $I$ and $J$ belong to the same $F$-component we can find two paths, one leading from $I$ to $J$ the other from $J$ to $I$, such that their respective weights $m$ and $m'$ satisfy $\delta(mm') = 0$. Let $n$ and $n'$ be the respective lengths of these paths. Note that

$$\beta^n r(I) = (A^n r)(I) = \sum_K A^n(I, K)\, r(K)$$

and that $mr(J)$ is an element of the sum on the right. Moreover every $r(K)_F$ is nonnegative by Perron-Frobenius theory [6], and we can conclude

$$\delta(r(I)) = \delta(\beta^n\, r(I)) \leq \delta(mr(J)).$$

From the equation $\beta^{n'} l(I) = (lA^{n'})(I)$ we similarly obtain

$$\delta(l(I)) \leq \delta(m' l(J)).$$

Hence,

$$\delta(l(I)r(I)) = \delta(l(I)) + \delta(r(I)) \leq \delta(mm') + \delta(l(J)r(J)) = \delta(l(J)r(J)).$$

Interchanging the roles of $I$ and $J$, we also have $\delta(l(J)r(J)) \leq \delta(l(I)r(I))$. ∎

For an $F$-component $A_{F,i}$, Lemma 1 allows us to pick any state $I$ of the component and define the **order** of $\delta(A_{F,i})$ on $D_0$ to equal $\delta(l(I)r(I))$. Let $\delta(A, F)$ be the minimum of $\delta(A_{F,i})$ taken over all principal $F$-components. We will see in Lemma 3 that $\delta(A_{F,j}) > \delta(A, F)$ whenever $A_{F,j}$ is a nonprincipal $F$-component. We first state our main result, which involves the (principal) $F$-components $A_{F,i}$ with $\delta(A_{F,i}) = \delta(A, F)$.

THEOREM: *Let $a = (a_1, \ldots, a_{k-1}) \in D$, and let $D_0$ be the connected component of $D$ containing $a$. Let $A_1, \ldots, A_n$ be the principal $F$-components whose order on $D_0$ is minimal.*

(i) *The limit*

$$\mu_{(A,a)}(C) = \lim_{(x_1,\ldots,x_k)\to(a_1,\ldots,a_{k-1},0^+)} \mu_{A(x_1,\ldots,x_k)}(C)$$

*exists for every cylinder set $C$ of $\Sigma_A$, and this defines a shift-invariant probability measure on (the shift of finite type underlying) $\Sigma_A$.*

(ii) *The support of $\mu_{(A,a)}$ is given by $A_1, \ldots, A_n$: It equals the (disjoint) union of the shifts of finite type $\Sigma_{A_i}$, $i = 1, \ldots, n$.*

(iii) *The restriction of $\mu_{(A,a)}$ to each $\Sigma_{A_i}$ is a constant multiple of the Markov measure $\mu_{A_i(a)}$. The constant in question, $\mu_{(A,a)}(\Sigma_{A_i})$, equals the value at $a$ of*

$$\frac{1}{T_F} \sum_{K \text{ state of } A_i} l(K)_F \, r(K)_F.$$

*(This expression depends only on $x_1, \ldots, x_{k-1}$ since $\delta(l(K)r(K)) = \delta(T)$ for every state $K$ of $A_i$.)*

A probability vector $p = (p_1, \ldots, p_n)$ is said to **refine** another probability vector $q = (q_1, \ldots, q_m)$ if there is a partition of $\{1, \ldots, n\}$ into $m$ sets $S_1, \ldots, S_m$ such that $\Sigma_{i \in S_j} p_i = q_j$. The following is a straightforward consequence of the theorem.

COROLLARY: *If $\phi$: $\Sigma_A \to \Sigma_B$ is a (wps-preserving) factor map, then we have $\mu_{(A,a)} \circ \phi^{-1} = \mu_{(B,a)}$. In particular, the probability vector*

$$(\mu_{(A,a)}(\Sigma_{A_1}), \ldots, \mu_{(A,a)}(\Sigma_{A_n}))$$

*refines the analogous probability vector for $\mu_{(B,a)}$.*

In preparation for the proof of the theorem, we first manipulate the matrix $A$. Using the fact that $l$ and $r$ are over $R[\beta] \subset \widetilde{R}[\widetilde{\beta}]$ and $\widetilde{\beta}$ is analytic at $a$, define a diagonal matrix $\Delta$ indexed by the states of $A$ and having $z^{\delta(l(I))}$ as its $I$-th diagonal entry. Replace $A$ by $\Delta A \Delta^{-1}$, and $l, r$ by $l\Delta^{-1}$, $\Delta r$. Note that the basic objects of our discussion, in particular $\beta$, wps of periodic orbits, $\delta(l(I)r(I))$ and the measures $\mu_{A(x_1,\ldots,x_k)}$, are not altered. We do, however, then have the convenience that$\delta(l(I)) = 0$ for every state $I$ of $A$, which we indicate by saying that $A$ is **left adapted**. This implies that $A$ is over $\mathbb{Z}[x_1^{\pm},\ldots,x_{k-1}^{\pm},z]$: Suppose an entry $A(I,J)$ has $\delta(A(I,J))<0$. Consider the equation

$$\beta\, l(J)=\sum_K l(K)A(K,J).$$

We have $\delta(\beta l(J))=0$ while $\delta(\sum_K l(K)A(K,J))<0$, which is absurd! So, the fact that $A$ is left adapted implies that it is over $\mathbb{Z}[x_1^{\pm},\ldots,x_{k-1}^{\pm},z]$.

Next we will recall from [7] the definition and properties of the face $A_F=A_{F,D_0}$ of $A$ corresponding to $F$ and $D_0$. Define row and column vectors $l_F$ and $r_F$ by setting $l_F(I)=l(I)_F$, $r_F(I)=r(I)_F$ for any state $I$ of $A$. It follows easily from Perron-Frobenius theory that each $l_F(I)$ and $r_F(I)$ is nontrivial and non-negative on $D_0\times\mathbb{R}^{++}$. It was observed in [7] that the proof of theorem 1 of [4] may be adapted to see that each $l_F(I)$, $r_F(I)$ is in fact positive on $D_0\times\mathbb{R}^{++}$, a property we shall make use of in the proof of the present theorem. We define $A_F$ by specifying the weighted subgraph $G(A_F)$ of $G(A)$ corresponding to $A_F$. For an edge $e$ of $G(A)$, let $I(e)$ and $J(e)$ denote its initial and terminal states, respectively. For each state $I$ we have

$$\beta\, r(I)=\sum_{I(e)=I}\mathrm{wt}_A(e)\, r(J(e)).$$

An edge $e$ belongs to $G(A_F)$ if and only if

$$\delta(\mathrm{wt}_A(e)\, r(J(e)))=\delta(\beta\, r(I(e))).$$

It is easy to see that $A_F$ has the same indexing set as $A$, no row of $A$ is trivial and $A_F\, r_F=\beta_F\, r_F$. In particular, $\beta_{A_F}=\beta_F$. It is also proved in [7] that the irreducible components of $A_F$ are precisely the $F$-components of $A$. Recall that an irreducible component of a nonnegative matrix is called a **sink** if we have $A(I,J)=0$ whenever $I$ is a state of the component and $J$ is not. Similarly, the component is called a **source** if $A(I,J)=0$ whenever $J$ is a state of the component and $I$ is not. It follows from the equation $A_F r_F=\beta_F r_F$ that the sinks of $A_F$ are precisely the principal $F$-components.

The definition of $A_F$ makes essential use of the right eigenvector of $A$. We could instead use in an analogous way the left eigenvector and, since $A$ is left adapted, we would then end up with the matrix obtained from $A$ by setting $y=0$. This matrix has the $F$-components of $A$ for its irreducible components and the principal $F$-components for sources. In particular, if $e_1 \cdots e_m$ is a path in $G(A)$ such that $J(e_m)$ lies in a principal $F$-component without $e_1 \cdots e_m$ being contained in this component then $y$ divides $\mathrm{wt}_A(e_1 \cdots e_m)$ in $\mathbb{Z}[x_1^{\pm}, \ldots, x_{k-1}^{\pm}, y]$. We record this as Lemma 2.

LEMMA 2: *If $e_1 \cdots e_m$ is a path of $G(A)$ such that the terminal state $J(e_m)$ lies in a principal $F$-component and $e_1 \cdots e_m$ is not contained in that component, then $\delta(\mathrm{wt}_A(e_1 \cdots e_m)) \geq 1$.*

LEMMA 3: *If $I$ is a state of a nonprincipal $F$-component of $A$ then*

$$\delta(l(I)\, r(I)) > \delta(A, F).$$

*Proof:* Note that, as $A$ is left adapted, $\delta(l(J)\, r(J)) = \delta(r(J))$ for every state $J$. Since $A_F\, r_F = \beta_F\, r_F$ and $r_F$ is positive (positivity at a single point of $D_0$ is all we need here), we can find a path $e_1 \cdots e_m$ in $G(A_F)$ such that $I(e_1) = I$ while $J = J(e_m)$ lies in a principal $F$-component. Then

$$\delta(r(I)) = \delta(r(I)\, \beta^m) = \delta(\mathrm{wt}_A(e_1 \cdots e_m)\, r(J)),$$

where the first equality follows from $\delta(\beta) = 0$ and the second from the definition of $A_F$. But, by Lemma 2, $\delta(\mathrm{wt}_A(e_1 \cdots e_m)) \geq 1$ and we conclude that

$$\delta(l(I)r(I)) = \delta(r(I)) > \delta(r(J)) = \delta(l(J)r(J)) \geq \delta(A, F). \qquad \blacksquare$$

*Proof of the Theorem:* Let $C = [e_1, \ldots, e_m]_h$ be a cylinder set. Put $I = I(e_1)$, $J = J(e_m)$ and

$$\alpha = \mathrm{wt}_A(e_1 \cdots e_m)\, l(I)\, r(J),$$

so that the $\mu_{A(x_1, \ldots, x_k)}$ measure of $C$ is the evaluation at $x_1, \ldots, x_k$ of $\alpha / T\beta^m$. Note that $\delta(T\beta^m) = \delta(T) = \delta(A, F)$ and that, since $A$ is left adapted, $\delta(l(I)) = 0$ and $\delta(\mathrm{wt}_A(e_1 \cdots e_m)) \geq 0$. If $J$ is not a state of $A_1, \ldots, A_n$ then, by the definition of $\delta(A, F)$ and Lemma 3,

$$\delta(\alpha) \geq \delta(r(J)) = \delta(l(J)r(J)) > \delta(A, F).$$

If $J$ is a state of one of $A_1, \ldots, A_n$ but $e_1 \cdots e_m$ is not contained in that component then, by Lemma 2, $\delta(\mathrm{wt}_A(e_1 \cdots e_m)) \geq 1$ and we have

$$\delta(\alpha) > \delta(r(J)) = \delta(A, F).$$

Thus, if $C$ is not contained in one of $\Sigma_{A_1}, \ldots, \Sigma_{A_n}$ then $\delta(T\beta^m) < \delta(\alpha)$ and the limit

$$(*) \qquad \lim_{(x_1,\ldots,x_k)\to(a_1,\ldots,a_{k-1},0+)} \mu_{A(x_1,\ldots,x_k)}(C)$$

exists and equals zero. Now suppose $C$ is contained in $\Sigma_{A_i}$ for some $1 \le i \le n$. Then $\delta(T\beta^m) = \delta(\alpha)$ and the limit $(*)$ is obtained by canceling the power of $z$ dividing the numerator of $\alpha/T\beta^m$ with that dividing the denominator and putting $(x_1, \ldots, x_{k-1}, z) = (a_1, \ldots, a_{k-1}, 0)$. This yields a positive number because, in this case, $\delta(\mathrm{wt}_A(e_1 \cdots e_m)) = 0$ and the entries of $l_F$ and $r_F$ are positive on $D_0 \times \mathbb{R}^{++}$ [4, 7]. This proves (i) and (ii).

For (iii) observe that, since $A_F r_F = \beta_F r_F$ and $A_i$ is a sink of $A_F$, the restriction of $r_F$ to the states of $A_i$ gives a right eigenvector of $A_i$ for $\beta_F = \beta_{A_i}$. Similarly, $l_F$ restricts to a left eigenvector of $A_i$. The $\mu_{A_i(a)}$ measure of the cylinder $C \subset \Sigma_{A_i}$ is then equal to the value at $a$ of

$$\frac{\mathrm{wt}_A(e_1 \cdots e_m)\, l_F(I)\, r_F(J)}{\beta_F^m \sum_{K \text{ state of } A_i} l_F(K)\, r_F(K)}.$$

On the other hand, the $\mu_{(A,a)}$ measure of $C$ is given by the value at $a$ of

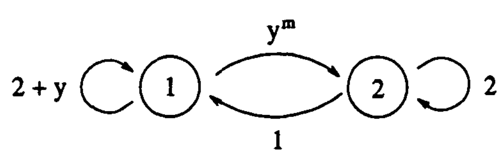
$$\frac{\mathrm{wt}_A(e_1 \cdots e_m)\, l_F(I)\, r_F(J)}{\beta_F^m T_F}. \qquad \blacksquare$$

Note that in this section we took $y = z^{d!}$ for specificity; in many cases a power lower than $d!$ may work. In particular, the examples of next section will have the property that $y = z$ works, so that there is no need to introduce the variable $z$.

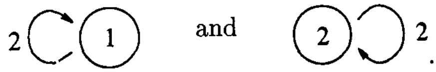
## 4. Examples

We will present matrices through their labeled directed graphs. Each edge of the graph will be labeled by an element of $R^+$, specifying the corresponding entry of the matrix. Each matrix will have its weight-per-symbol polytope contained in the half-space $y \ge 0$, and we will be concerned with the face $F$ lying in the hyperplane $y = 0$.

*Example 1:* Let $m$ be an even integer and consider the following graph:



Then $\beta$ satisfies $\beta^2 - (4+y)\beta + 4 + 2y - y^m = 0$ and, with the given ordering of states, has left and right eigenvectors $l = (\beta - 2, y^m)$ and $r = (\beta - 2, 1)^t$. There are two $F$-components, both principal:




Postponing the discussion of the case $m = 2$, we assume $m > 2$. Then $\beta$ is analytic at $y = 0$ and, near $y = 0$,

$$\beta = 2 + y + O(y^2).$$

Since

$$\delta(l(1)r(1)) = \delta((\beta-2)^2) = \delta(y^2 + O(y^3)) = 2$$

and

$$\delta(l(2)r(2)) = \delta(y^m) = m > 2,$$

the boundary measure at the point $y = 0$ is supported by the first principal $F$-component and is simply the measure of maximal entropy on this two-shift.

*Example 2:* Now take $m = 2$ in the graph above, and let $B$ be the corresponding matrix. Then $\beta = \beta_B = 2 + \rho y$ where $\rho = \frac{1}{2}(1+\sqrt{5})$ is the golden mean. Let $B_1, B_2$ be the $F$-components of $B$ corresponding to states 1 and 2, respectively. Since we now have

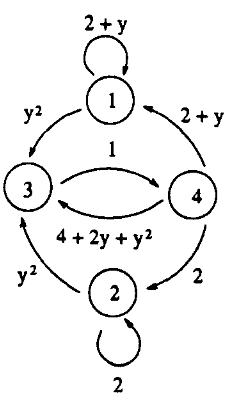
$$\delta(l(1)r(1)) = \delta(l(2)r(2)) = 2,$$

the boundary measure at $y = 0$ is supported by $\Sigma_{B_1} \cup \Sigma_{B_2}$. Moreover,

$$l(1)_F\, r(1)_F = (\beta - 2)_F^2 = \rho^2 y^2 = (1+\rho)y^2,$$
$$l(2)_F\, r(2)_F = y^2,$$

so that $\Sigma_{B_1}, \Sigma_{B_2}$ have measure $\frac{1+\rho}{2+\rho}$, $\frac{1}{2+\rho}$, respectively.

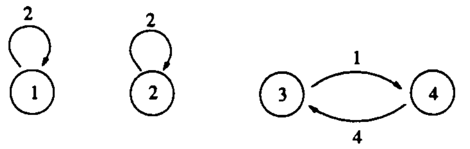
Next let $A$ be the matrix corresponding to the following graph:



Then $\beta_A = \beta = \beta_B = 2 + \rho y$ and, with the given ordering of states, the corresponding left and right eigenvectors are:

$$l = l_A = ((2+y)(\beta-2), 2(\beta-2-y), y^2\beta, y^2),$$
$$r = r_A = (\beta-2, \beta-2-y, 1, \beta)^t.$$

The matrix $A$ three $F$-components $A_1, A_2, A_3$, corresponding to the graphs

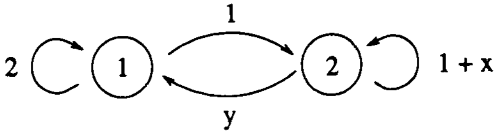



All three $F$-components are principal,

$$\delta(A,F) = \delta(A_1) = \delta(A_2) = \delta(A_3) = 2,$$

and an easy calculation shows that the boundary measure at $y = 0$ assigns measure $\frac{\rho+1}{5}$, $\frac{2-\rho}{5}$, $\frac{2}{5}$ to $\Sigma_{A_1}$, $\Sigma_{A_2}$, $\Sigma_{A_3}$. Observe that the probability vector $\left(\frac{\rho+1}{5}, \frac{2-\rho}{5}, \frac{2}{5}\right)$ does not refine $\left(\frac{1+\rho}{2+\rho}, \frac{1}{2+\rho}\right)$; a quick way of seeing this is to check that no entry of the first vector is equal to an entry of the second. In view of the corollary in section 3, this shows that there is no (wps-preserving) factor map of $\Sigma_A$ onto $\Sigma_B$. We also remark that consideration of periodic points (zeta functions) reveals that there in no factor map of $\Sigma_B$ onto $\Sigma_A$, even if we drop the requirement that the map be wps-preserving. On the other hand we can find a nontrivial matrix $S$ over $R^+$ satisfy ng $AS = SB$. In particular such a matrix can be constructed from the eigenvectors. By [5], it follows that $\Sigma_A$ and $\Sigma_B$ have a common extension by wps-preserving factor maps.

*Example 3:* In order to define the boundary measure at a point, we needed $\beta$ to be analytic at that boundary point. In this example we indicate what can happen if $\beta$ is not analytic at the boundary point. Let $A$ be defined by the following graph:




Then there are two $F$-components, with matrices $A_1 = [2]$ and $A_2 = [1+x]$, so that

$$\beta_F(x) = \begin{cases} 2 & \text{if } 0 < x \leq 1, \\ 1+x & \text{if } x \geq 1. \end{cases}$$

The eigenvectors of $A$ are $l = (y, \beta - 2)$ and $r = (1, \beta - 2)^t$. Clearly $\beta$ is not analytic at $(x, y) = (1, 0)$. But it is analytic at every point $(a, 0)$ with $a > 0$, $a \neq 1$. For $0 < a < 1$ we have $\delta(A_1) = \delta(y) = 1$ and, since $\beta(a, 0) = 2$ for $0 < a < 1$, we also have $\delta(A_2) = \delta((\beta - 2)^2) \geq 2$. Thus, in this case $\mu_{(A,a)}$ is supported by $\Sigma_{A_1}$. On the other hand, for $a > 1$ we have $\delta(A_1) = 1$, $\delta(A_2) = \delta((x - 1)^2) = 0$ which means that $\mu_{(A,a)}$ is supported by $\Sigma_{A_2}$. So, the measure (even the support) we get depends on the path along which we approach $(1, 0)$. This situation is reminiscent of a "phase transition".

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